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An estimate of the error of the linearized equations of motion of an axisymmetric projectile☆

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ABSTRACT

An estimate is obtained, by small-parameter methods, of the error of the solution for the system of equations of motion of a projectile in the atmosphere, linearized for small angles of attack with respect to the variables describing the angular oscillations of the axis of symmetry, compared with the solution of the initial non-linear system for the same initial conditions.

It is not difficult to obtain such an estimate by numerical integration. However, it derivation in analytical form is necessary in order to formulate the conditions for which linearization of the equations of motion of the projectile is admissible, to reveal the main factors which determine the value of the error of the linearized equations, and to analyse the mechanism by which the error arises.

1. Initial notation

We will consider the motion of an axisymmetric projectile in a gravity field under the action of the system of aerodynamic forces and moments usually assumed in ballistics. We will use the following variables to describes the motion: x, y, z are the coordinates of the centre of mass of the projectile in the launching system of Cartesian coordinates $Oxyz$ (the Ox axis is directed horizontally in the firing direction, while the Oy axis is directed vertically upwards), ν, θ, ψ are the components of the vector $\boldsymbol{\nu}$ of the velocity of the centre of mass (θ is the angle between the Ox axis and the projection of $\boldsymbol{\nu}$ onto the Oxy plane, and ψ is the angle between this projection and $\boldsymbol{\nu}$), α and β are the projections of the unit vector of the axis of symmetry of the projectile onto the axis of the semi-velocity system of coordinates orthogonal to $\boldsymbol{\nu}$, and p, q, r are the projections of the vector $\boldsymbol{\omega}$ of the angular velocity of the projectile onto the axis of the semi-connected system of coordinates. We will denote by I_1, I_2 and m the axial and equatorial central moments of inertia of the projectile and its mass, and g is the acceleration due to gravity.

Suppose R_x is the drag, R_y is the lift, R_z is the Magnus force, M_y is the Magnus moment, and M_z is the tilting moment. The projection of the damping moment vector onto the longitudinal axis is represented in the form $M_{\mathrm{p}}p$, and its projection onto the transverse axes of the semi-connected system of coordinates are equal to $M_\Omega q$, M_Ω.

As is well known, the quantities R_x, M_p and M_Ω are even functions of the angle of attack δ while R_y, R_z, M_y and M_z are odd functions of the angle of attack, which all depend on y, ν, while R_z and M_y also depend on p. The aerodynamic forces and moments R_x, R_y, R_z, M_y and M_z are approximately proportional to ν^2, while the coefficients M_p and M_Ω of the damping moments are approximately proportional to ν. Hence, separating the factors ν and ν^2 in the expressions for the corresponding forces and moments, we obtain

$$R_x = \upsilon^2 R_1, \quad R_y = \upsilon^2 R_2 \sin\delta, \quad R_z = \upsilon^2 R_3 \sin\delta$$
$$M_y = \upsilon^2 M_2 \sin\delta, \quad M_z = \upsilon^2 M_3 \sin\delta, \quad M_p = \upsilon M_{1D}, \quad M_\Omega = \upsilon M_{2D}$$

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The quantities $R_1, R_2, R_3, M_2, M_3, M_{1D}, M_{2D}$ depend on y and v, while R_3 and M_2 also depend on p, and all these quantities are even functions of the angle δ, and change comparatively weakly when v changes over the flight range.

In the case of practical interest, when $|\delta| \le \pi/2$, the mapping $\delta \to \sin\delta$ is one-to-one. Hence, the new functions introduced can be assumed to depend on δ by means of the quantity $\zeta \to \sin\delta$. Since these functions are even in δ, they are also even in ζ and consequently can be represented by expansions in even powers of ζ, i.e., in powers of $\zeta = \alpha + \beta^2$.

2. The equations of motion, formulation of the problem

We will use a normalization procedure[1] to introduce a small parameter into the equations of motion of the projectile. It is based on the fact that upper characteristic values of their moduli are chosen as the new scales of the phase variables and the functions which depend on them, i.e., values close to the maxima in time for all the trajectories of the systems of the class considered. Below numerical normalization is employed using a decimal numerical scale: we will choose decimal numerical orders of the corresponding characteristic quantities as the new scales.

Using square brackets for the notation of the new scales, we have

$$[x] = [y] = 10^4 \text{ m}, \quad [z] = 10^2 \text{ m}, \ [v] = 10^3 \text{ ms}^{-1}, \ [\theta] = 1, \ [\psi] = 10^{-2}$$

$$[p] = 10^3 \text{ s}^{-1}, \quad [q] = [r] = 1 \text{ s}^{-1}, \quad [\alpha] = [\beta] = 10^{-2}, \quad [R_1/m] = 10^{-5} \text{ m}^{-1}$$

$$[R_2/m] = [R_3/m] = 10^{-4} \text{ m}^{-1}, \quad [M_{1D}/I_1] = 10^{-5} \text{ m}^{-1}$$

$$[M_{2D}/I_2] = 10^{-3} \text{ m}^{-1}, \quad [M_2/I_2] = 10^{-4} \text{ m}^{-2}, \quad [M_3/I_2] = 10^{-2} \text{ m}^{-2}$$

We will take 10^{-2} s as the unit of time, while the time of flight of the projectile from the instant it is shot t_0 to the instant t_1 when it falls to Earth when firing the maximum range is measured in tens of seconds. The value of p does not change very much during the flight whereas v^2 changes from values of the order of $[v]^2$ on the initial part of the trajectory to values of the order of 10^{-1} $[v]^2$, which, for large firing angles are reached close to the peak of the trajectory and on its descending part. The small parameter ε is introduced instead of the number 10^{-1}.

Henceforth the symbols $O(\varepsilon^n)$ $[O^*(\varepsilon^n)]$ denote functions of the phase variables, the time and the parameter ε, which, in the range of variation of the phase variables and the time considered have an order no less than ε^n [equal to ε^n] as $\varepsilon \to 0$; for positive functions the notation $O_+(\varepsilon^n)$ $[O_+^*(\varepsilon^n)]$ is employed.

Since the chosen scales for the variables $x, y, z, v, \theta, \psi, p$ are equal to decimal orders of the maximum possible values of their moduli, taking into account the remarks regarding the ranges of variation of v and p, after introducing the small parameter, the vector $\xi = (x, y, \varepsilon^2 z, v, \theta, \varepsilon^2\psi, p)$ belongs to the parallelepiped

$$\Xi = \{\xi : (0, 0, -\varepsilon^2 C_z^*, \sqrt{\varepsilon} C_{v*}, -C_\theta^*, -\varepsilon^2 C_\psi^*, C_{p*}) \le \xi \le \le (C_x^*, C_y^*, \varepsilon^2 C_z^*, C_v^*, C_\theta^*, \varepsilon^2 C_\psi^*, C_p^*)\} \tag{2.1}$$

on all the flight trajectories. Here positive constants of the order of unity are denoted by the letter C with subscripts. The vectors $\xi^{(5)} = (y, v, \theta, \varepsilon^2\psi, p)$, $\xi^{(3)} = (y, v, p)$ belong to corresponding parallelepipeds $\Xi^{(5)}$, $\Xi^{(3)}$. The moduli of the new variables q, r, α and β, are limited to values of the order of unity for additional conditions (see Section 3).

As regards the numerical orders of the terms of the aerodynamic functions that are non-linear in $\zeta = \sin\delta$, we make assumptions which, after introducing the small parameter, lead to the relations

$$R_j(y, v, \zeta, \varepsilon) = R_j^{(0)}(y, v) + \varepsilon^3\zeta^2 R_j^{(2)}(y, v, \zeta, \varepsilon), \quad j = 1, 2$$

$$R_3(y, v, p, \zeta, \varepsilon) = R_3^{(0)}(y, v, p) + \varepsilon^3\zeta^2 R_3^{(2)}(y, v, p, \zeta, \varepsilon)$$

$$M_{1D}(y, v, \zeta, \varepsilon) = M_{1D}^{(0)}(y, v) + \varepsilon^3\zeta^2 M_{1D}^{(2)}(y, v, \zeta, \varepsilon)$$

$$M_{2D}(y, v, \zeta, \varepsilon) = M_{2D}^{(0)}(y, v) + \varepsilon^2\zeta^2 M_{2D}^{(2)}(y, v, \zeta, \varepsilon)$$

$$M_2(y, v, p, \zeta, \varepsilon) = M_2^{(0)}(y, v, p) + \varepsilon^2\zeta^2 M_2^{(2)}(y, v, p, \zeta, \varepsilon)$$

$$M_3(y, v, \zeta, \varepsilon) = M_3^{(0)}(y, v) + \varepsilon^4\zeta^2 M_3^{(2)}(y, v, \zeta, \varepsilon)$$

We will write the equations of motion of the projectile, containing the small parameter ε, separating terms that are linear in the variables, q, r, α and β, and additional non-linear terms. As a result we obtain the following equations of translational motion and longitudinal rotation

$$\begin{aligned}
&\dot{x} = \varepsilon^3 \upsilon\cos\theta\cos\varepsilon^2\psi, \quad \dot{y} = \varepsilon^3\upsilon\sin\theta\cos\varepsilon^2\psi, \quad \varepsilon^2\dot{z} = \varepsilon^3\upsilon\sin\varepsilon^2\psi \\
&\dot{\upsilon} = \varepsilon^3\frac{\upsilon^2}{m}R_1^{(0)}(y, \upsilon) - \varepsilon^4 g\sin\theta\cos\varepsilon^2\psi + h_\upsilon(y, \upsilon, \alpha, \beta, \varepsilon) \\
&\dot{\theta} = -\varepsilon^4\frac{g\cos\theta}{\upsilon\cos\varepsilon^2\psi} + \varepsilon^4\frac{\upsilon}{m\cos\varepsilon^2\psi}[R_2^{(0)}(y, \upsilon)\alpha - R_3^{(0)}(y, \upsilon, p)\beta] + h_\theta(y, \upsilon, \psi, p, \alpha, \beta, \varepsilon)\,(2.2) \\
&\varepsilon^2\dot{\psi} = \varepsilon^4\frac{g}{\upsilon}\sin\theta\sin\varepsilon^2\psi + \varepsilon^4\frac{\upsilon}{m}[R_3^{(0)}(y, \upsilon, p)\alpha + R_2^{(0)}(y, \upsilon)\beta] + h_\psi(y, \upsilon, p, \alpha, \beta, \varepsilon) \\
&\dot{p} = \varepsilon^4\frac{p}{I_1}\upsilon M_{1D}^{(0)}(y, \upsilon) + h_p(y, \upsilon, p, \alpha, \beta, \varepsilon)
\end{aligned} \tag{2.2}$$

and the following equations of angular oscillations of the axis of symmetry

$$\begin{aligned}
&\dot{\Omega} = a(y, \upsilon, p, \varepsilon)\Omega + b(y, \upsilon, p, \varepsilon)\Delta + h_\Omega(y, \upsilon, \theta, \psi, p, q, r, \alpha, \beta, \varepsilon) \\
&\dot{\Delta} = -i\Omega - k(y, \upsilon, p, \varepsilon)\Delta + l(\upsilon, \theta, \varepsilon^2\psi, \varepsilon) + h_\Delta(y, \upsilon, \theta, \psi, p, q, r, \alpha, \beta, \varepsilon)
\end{aligned} \tag{2.3}$$

In the linear part of Eqs. (2.3) the complex variables $\Omega = q + ir$, $\Delta = \alpha + i\beta$ are used and also the notation

$$\begin{aligned}
&a = [\varepsilon^2\upsilon M_{2D}^{(0)} + iI_1 p]/I_2, \quad b = \upsilon^2[\varepsilon^2 M_2^{(0)} + iM_3^{(0)}]/I_2 \\
&k = \varepsilon^2\upsilon[R_2^{(0)} + iR_3^{(0)}]/m, \quad l = \varepsilon^2 g(\cos\theta - i\sin\theta\sin\varepsilon^2\psi)/\upsilon
\end{aligned} \tag{2.4}$$

When $\xi \in \Xi$ and $q, r, \alpha, \beta = O(1)$ the terms in Eqs. (2.2) and (2.3), non-linear in the variables q, r, α and β, are as follows:

$$h_\theta, h_\psi, h_p = O(\varepsilon^7); \quad h_\upsilon = O(\varepsilon^6); \quad h_\Omega, h_\Delta = O(\varepsilon^4) \tag{2.5}$$

In formulae (2.2), (2.4) and later, the functions denoted by capital Latin letters for $\xi \in \Xi$ are equal to $O(1)$ together with their partial derivatives with respect to the components ξ. System (2.2), (2.3) is considered in the time interval $[t_0, t_1]$ of length $t_1 - t_0 = O(\varepsilon^{-3})$.

Equations (2.2) can be written in vector form as

$$\dot{\xi} = f(\xi, \varepsilon) + f_\alpha(\xi, \varepsilon)\alpha + f_\beta(\xi, \varepsilon)\beta + h_\xi(\xi, \alpha, \beta, \varepsilon) \tag{2.6}$$

The definitions of f, f_α and f_β follow from a comparison of Eqs. (2.2) and (2.6).

The introduction of a small parameter into the equations of motion of the projectile is possible because the time constant for angular oscillations of the axis of symmetry is considerably less than the time constant for translational motion.

The system of equations of motion of the projectile, linearized with respect to the angular motion variables q, r, α and β (the L-system), is obtained from the initial system (2.2), (2.3) by dropping non-linear terms, denoted by the letter h with subscripts. The L-system is widely used both in theoretical research on ballistics and in practical calculations.[2,3]

Suppose $\xi, \Omega, \Delta(t, \varepsilon)$ is the solution of the initial system (2.2), (2.3) for initial conditions specified at the instant of firing t_0. We denote by $\xi_L, \Omega_L, \Delta_L(t, \varepsilon)$ the solution of the L-system for the same initial conditions. The main problem of this paper is to estimate the error of the solution of the L-system, namely, to determine the orders with respect to ε of the quantities $||\xi(t, \varepsilon) - \xi_L(t, \varepsilon)||$, $|\Omega(t, \varepsilon) - \Omega_L(t, \varepsilon)|$, $|\Delta(t, \varepsilon) - \Delta_L(t, \varepsilon)|$ when $t \in [t_0, t_1]$. Here $||\xi||$ is the norm of the vector ξ, equal to the maximum of the moduli of its components.

The solution of this problem is constructed in three stages. In the first (preliminary) stage, approximate solutions of the equations of angular motion for the initial system and the L-system are found. They are determined as follows. If we consider the relation $\xi(t, \varepsilon)$ as known, the coefficients of the linearized equations of the angular motion will be known functions of t and ε. This enables us to obtain (using the asymptotic method) an approximate solution of the linearized equations of angular motion, which is then considered as an approximate solution of the non-linear equations of angular motion (2.3). In this solution estimates of the error for the complex amplitudes are established by comparison with the complex amplitudes and the exact solution $\Omega, \Delta(t, \varepsilon)$ of Eqs. (2.3). For the known relation $\xi_L(t, \varepsilon)$ an approximate solution of the linearized equations of angular motion is constructed in exactly the same way, and estimates of the error of the complex amplitudes are derived by comparison with the complex amplitudes in the exact solution $\xi_L, \Delta_L(t, \varepsilon)$ of these equations.

At the second stage, the differential equations for $\xi, \xi_L(t, \varepsilon)$ are written in integral form, the quantities $\Delta, \Delta_L(t, \varepsilon)$ are expressed in terms of the complex amplitudes, and, using the estimates obtained for the complex amplitudes, the required estimate of the error of the variables ξ_L is derived. Using it, it is easy to estimate, at the third stage, the errors of the variables Ω_L, Δ_L.

3. The flight accuracy conditions

Suppose

$$w = (a - k)^2/4 - ib + ak - \dot{a}_0/2 \tag{3.1}$$

where $\dot{a}_0(\xi^{(3)}, \varepsilon) = i\varepsilon^4 p\nu M_{1D}^{(0)}(y, \nu)/I_2$ is the principal term in the expression for the derivative of the function $a(\xi^{(3)}, \varepsilon)$ by virtue of Eqs. (2.2). Substituting Eqs. (2.4) into (3.1), we obtain

$$w(\xi^{(3)}, \varepsilon) = -\frac{p^2 I_1^2}{4I_2^2}\sigma^2(y, \upsilon, p) + \varepsilon^2 W^{(2)}(y, \upsilon, p, \varepsilon) \tag{3.2}$$

Here

$$\sigma^2(y, \upsilon, p) = 1 - 4\upsilon^2 M_3^{(0)}(y, \upsilon) I_2/(p^2 I_1^2) \tag{3.3}$$

The projectile and the gun are constructed in such a way as to ensure that expression (3.3) is positive at the instant of firing t_0, in which case $0.6 < \sigma(t_0) < 0.7$, $\nu(t_0) = O^*(1)$. On the rising part of the trajectory the velocity ν decreases, and on the middle part of the trajectory and on its descending branch the velocity may reach values of the order of $\varepsilon^{1/2}$. The value of p is equal to $O_+^*(1)$ during the whole flight time of the projectile. Hence, over the whole flight trajectory the Maievskii condition $\sigma^2 > 0$ is satisfied, and the gyroscopic stability factor σ lies in the limits

$$\sigma(t_0) \le \sigma \le 1 - O_+(\varepsilon); \quad \sigma(t_0) = O_+^*(1) \tag{3.4}$$

At the same time $w = O*(1)$ always.

We will agree $\sqrt{w} = w^{1/2}$ to mean only one value of the square root of w, equal to $\sqrt{w} = i\sqrt{_+(-w)}$, where $\sqrt{_+}$ is the principal value.[4]

We will introduce new functions by the formulae

$$\lambda_j(\xi^{(3)}, \varepsilon) = \frac{a-k}{2} \pm \sqrt{w}, \quad \lambda_j^+(\xi^{(5)}, \alpha, \beta, \varepsilon) = \lambda_j - \frac{\dot{w}}{4w}; \quad j = 1, 2 \tag{3.5}$$

where the upper sign corresponds to $j = 1$ and lower sign corresponds to $j = 2$. The derivative $w(\xi^{(5)}, \alpha, \beta, \varepsilon)$ of $w(\xi^{(5)}, \varepsilon)$ is calculated by virtue of subsystem (2.2). Using formulae (2.4), (3.2) and (3.4) we obtain $\lambda_j = ip(1 \pm \sigma)I_1/(2I_2) + O(\varepsilon^2)$ and, assuming $\lambda_j = n_j + i_j$ $(j = 1,2)$, we obtain approximate expressions for the frequencies

$$\omega_j = p(1 \pm \sigma)I_1/(2I_2) + O(\varepsilon^2)$$

Hence, taking inequalities (3.4) into account, we obtain estimates in $\Xi^{(3)}$

$$\begin{gathered} w, \lambda_1, \omega_1 = O^*(1); \quad O^*(\varepsilon) \le \lambda_2, \omega_2 \le O^*(1); \quad n_1, n_2 = O(\varepsilon^2) \\ \lambda_1 - \lambda_2, \omega_1 - \omega_2 = O^*(1); \quad \lambda_1^{-1}, \omega_1^{-1} = O^*(1); \quad \lambda_2^{-1}, \omega_2^{-1} = O(\varepsilon^{-1}) \end{gathered} \tag{3.6}$$

Suppose that, for the solution ξ, Ω, Δ (t, ε) of Eqs. (2.2), (2.3) at the instant of firing, the conditions Ω, Δ $(t_0, \varepsilon) = O(1)$ are satisfied, and along the flight trajectory of the projectile when $\xi = \xi(t, \varepsilon)$ the Maievskii inequality $\sigma^2 > 0$, inequality (3.4) and also the inequalities n_1, $n_2 \le O_+(\varepsilon^4)$ are satisfied. Then[5]

$$\Omega, \Delta(t, \varepsilon) = O(1), \quad t \in [t_0, t_1] \tag{3.7}$$

The above conditions ensure "accuracy" of the flight of the projectile, i.e. (taking into account the new scales introduced for α and β), that the angle of attack δ is small. The relations $\xi \in \Xi$ and Ω, $\Delta = O(1)$ enable the orders with respect to ε to be found for the functions of the phase coordinates of system (2.2) and (2.3).

4. The approximate solution of the equations of angular motion

We will denote by e, d $(\xi)^{(5)}, \varepsilon)$ the values of Ω and Δ for which the right-hand sides of Eqs. (2.3) vanish when h_Ω, h_Δ are dropped

$$e = bl/(ib - ak), \quad d = -al/(ib - ak) \tag{4.1}$$

We introduce the function

$$\rho(\xi^{(5)}, q, r, \alpha, \beta, \varepsilon) = \frac{1}{2w^{1/2}}\left(\frac{\ddot{w}}{4w} - \frac{5\dot{w}^2}{16w^2} - \frac{\dot{a}_1 + \dot{k}}{2}\right) \tag{4.2}$$

Considering the relation ξ (t, ε) for the exact solution of Eqs. (2.2), (2.3) as known, we define the approximate quasisolution of the equations of angular motion (2.3) by the formulae

$$\begin{aligned} \tilde{\Omega}^+(t, \varepsilon) &= i\sum_{j=1}^{2} [\lambda_j^+(t, \varepsilon) + k(t, \varepsilon)]\tilde{s}_j^+(t, \varepsilon)e^{i\varphi_j(t, \varepsilon)} + e(t, \varepsilon) \\ \tilde{\Delta}^+(t, \varepsilon) &= \sum_{j=1}^{2} \tilde{s}_j^+(t, \varepsilon)e^{i\varphi_j(t, \varepsilon)} + d(t, \varepsilon) \end{aligned} \tag{4.3}$$

where

$$\tilde{s}_j^+(t,\varepsilon) = C_j^+ \exp\int\limits_{t_0}^{t}\left[n_j(\tau,\varepsilon) - \frac{\dot{w}(\tau,\varepsilon)}{4w(\tau,\varepsilon)}\right]d\tau$$

$$\varphi_j(t,\varepsilon) = \int\limits_{t_0}^{t}\omega_j(\tau,\varepsilon)d\tau; \quad j = 1,2 \tag{4.4}$$

The complex constants C_j^+ are determined by the initial conditions

$$\tilde{\Omega}^+, \tilde{\Delta}^+(t_0,\varepsilon) = \Omega, \Delta(t_0,\varepsilon)$$

Here and henceforth the corresponding functions (2.4), (4.1) of arguments ξ, ε, calculated for $\xi = \xi(t,\varepsilon)$ are denoted by $k(t,\varepsilon)$, $e(t,\varepsilon)$, while the equalities (3.5) and (4.2) of the function of the phase variables and ε, calculated for the solution ξ, Ω and Δ (t,ε) of system (2.2) and (2.3), are denoted by $\lambda_j^+(t,\varepsilon)$, $\rho(t,\varepsilon)$.

It follows from (4.4) that

$$\tilde{s}_j^+(t,\varepsilon) = C_j^+\frac{w^{1/4}(t_0,\varepsilon)}{w^{1/4}(t,\varepsilon)}\exp\int\limits_{t_0}^{t}n_j(\tau,\varepsilon)d\tau, \quad j = 1,2$$

Since w and n_j depend only on t here via $\xi^{(3)}(t,\varepsilon)$, the functions $\tilde{s}_j^+(t,\varepsilon)$ in Eqs. (4.3) are determined if we know $\xi(t,\varepsilon)$. However, formulae (4.3) for $\tilde{\Omega}^+$ contain the quantities $\lambda_j^+(t,\varepsilon)$, which are expressed in terms of the unknown functions α and $\beta(t,\varepsilon)$. Hence, the approximation solution of Eqs. (2.3), defined by formulae (4.3), is called a quasisolution.

We will introduce the variables s_1^+, s_2^+ (complex amplitudes) by the formulae

$$\Omega = i\sum_{j=1}^{2}[\lambda_j^+(t,\varepsilon) + k(t,\varepsilon)]s_j^+e^{i\varphi_j(t,\varepsilon)} + e(t,\varepsilon)$$

$$\Delta = \sum_{j=1}^{2}s_j^+e^{i\varphi_j(t,\varepsilon)} + d(t,\varepsilon) \tag{4.5}$$

We will show that when $t \in [t_0, t_1]$ the following relations hold

$$s_j^+(t,\varepsilon) - \tilde{s}_j^+(t,\varepsilon) = O(\varepsilon), \quad ds_j^+(t,\varepsilon)/dt - d\tilde{s}_j^+(t,\varepsilon)/dt = O(\varepsilon^3); \quad j = 1,2 \tag{4.6}$$

We will change to the variables s_1^+, s_2^+ in Eqs. (2.3). We obtain for s_1^+, s_2^+ a system of two differential equations, close to linear diagonal. After conversion to integral form it becomes

$$s_j^+(t,\varepsilon) - \tilde{s}_j^+(t,\varepsilon) = h_{1j}(t,\varepsilon) + h_{2j}(t,\varepsilon) + h_{3j}(t,\varepsilon), \quad j = 1,2 \tag{4.7}$$

where

$$h_{1j}(t,\varepsilon) = \pm\int\limits_{t_0}^{t}\rho(s_1^+e^{i\varphi_1} + s_2^+e^{i\varphi_2})e^{-i\varphi_j}\left[\exp\int\limits_{\tau}^{t}\left(n_j - \frac{\dot{w}}{4w}\right)d\tau_1\right]d\tau$$

$$h_{2j}(t,\varepsilon) = \mp\int\limits_{t_0}^{t}\frac{ih_\Omega + (\lambda_{3-j}^+ + k)h_\Delta}{2w^{1/2}}e^{-i\varphi_j}\left[\exp\int\limits_{\tau}^{t}\left(n_j - \frac{\dot{w}}{4w}\right)d\tau_1\right]d\tau$$

$$h_{3j}(t,\varepsilon) = \pm e^{-i\varphi_j(t,\varepsilon)}\int\limits_{t_0}^{t}\frac{i\dot{e} + (\lambda_{3-j}^+ + k)\dot{d}}{2w^{1/2}}\left[\exp\int\limits_{\tau}^{t}\lambda_j^+d\tau_1\right]d\tau \tag{4.8}$$

The arguments τ, ε and τ_1, ε under the integral signs are omitted here.

The following estimates hold for the functions (4.8) when $t \in [t_0, t_1]$

$$h_{1j}(t,\varepsilon) = O(\varepsilon^2), \quad h_{2j}(t,\varepsilon) = O(\varepsilon), \quad h_{3j}(t,\varepsilon) = O(\varepsilon^2) \tag{4.9}$$

In fact, since $n_1, n_2 \le O_+(\varepsilon^4)$ according to the assumptions in Section 3, then when $t \in [t_0, t_1]$ the exponents of the integrals in formulae (4.8) are close to unity. Inverting the replacement formulae (4.5) taking relations (3.7) into account we obtain $s_j^+ = O(1)$. Since $w = O^*(1)$, $\dot{w} = O(\varepsilon^3)$, $\ddot{w} = O(\varepsilon^6)$, $\dot{a}_1 = O(\varepsilon^5)$, $\dot{k} = O(\varepsilon^5)$ we have $\rho = O(\varepsilon^5)$. Hence, in the first equality of (4.8) the integrand is equal to $O(\varepsilon^5)$. In the second equality of (4.8) the integrand (according to formulae (2.5), is equal to $O(\varepsilon^4)$. Hence the first two estimates of (4.9) follow when $t - t_0 = O(\varepsilon^{-3})$.

Turning to the third estimate we note that the right-hand sides of Eqs. (2.2) and formulae (2.4) are represented in the form of sums, each term of which is equal to the product of a factor of the form $\varepsilon^{m_1}\nu^{m_2}$ and a function of the order of unity. Consequently, the order of each term is determined by the order of the velocity ν when it changes in the range $O_+^*\varepsilon^{1/2} \le \nu \le O_+^*(1)$ considered. Hence, after substituting expressions (2.4) into equalities (4.1), the latter can be written in the form

$$e(\xi^{(5)}, \varepsilon) = \frac{\varepsilon^2}{\upsilon}E(\xi^{(3)}, \varepsilon)(\cos\theta - i\sin\theta\sin\varepsilon^2\psi)$$
$$d(\xi^{(5)}, \varepsilon) = \frac{\varepsilon^2}{\upsilon^3}D(\xi^{(3)}, \varepsilon)(\cos\theta - i\sin\theta\sin\varepsilon^2\psi) \tag{4.10}$$

The functions E and D are equal to $O^*(1)$, and their partial derivatives with respect to the components $\xi^{(3)}$ are equal to $O(1)$ when $\xi^{(3)} \in \Xi^{(3)}$. Differentiation of equalities (4.10) with respect to t, by virtue of Eqs. (2.2), leads to the following expressions

$$\dot e(\xi^{(5)}, \alpha, \beta, \varepsilon) = \dot e^{(0)}(\xi^{(5)}, \varepsilon) + \dot e^{(1)}(\xi^{(5)}, \alpha, \beta, \varepsilon)$$
$$\dot d(\xi^{(5)}, \alpha, \beta, \varepsilon) = \dot d^{(0)}(\xi^{(5)}, \varepsilon) + \dot d^{(1)}(\xi^{(5)}, \alpha, \beta, \varepsilon) \tag{4.11}$$

in which

$$\dot e^{(0)}(\xi^{(5)}, \varepsilon) = \varepsilon^5 E_{50}(\xi^{(5)}, \varepsilon) + \varepsilon^6\upsilon^{-2}E_{6,-2}(\xi^{(5)}, \varepsilon)$$
$$\dot d^{(0)}(\xi^{(5)}, \varepsilon) = \varepsilon^5\upsilon^{-2}D_{5,-2}(\xi^{(5)}, \varepsilon) + \varepsilon^6\upsilon^{-4}E_{6,-4}(\xi^{(5)}, \varepsilon) \tag{4.12}$$

while the following estimates hold for $\dot e^{(1)}, \dot d^{(1)}$

$$\dot e^{(1)} = O(\varepsilon^6), \quad \dot d^{(1)} = O(\varepsilon^5) \tag{4.13}$$

from which it follows that when $t \in [t_0, t_1]$ the contribution to h_{3j} of the second terms on the right-hand sides of formulae (4.11) is equal to $O(\varepsilon^2)$. The contribution of the terms $\dot e^{(0)}, \dot d^{(0)}$, which depend on t only through $\xi^{(5)}$ (t, ε), is equal to $O(\varepsilon^2)$ when $j = 1$ and $O(\varepsilon^3)$ when $j = 2$. In order to establish this it is sufficient to write the functions λ_j in the form $\lambda_1 = \Lambda_1$, $\lambda_2 = \nu^2\Lambda_2$ where $\Lambda_1, \Lambda_2 = O^*(1)$, use equalities (4.12) and integrate by parts, separating out $d(\exp)\int_\tau^t \lambda_j^+ d\tau_1)$. Thus the third equality of (4.9) has been proved.

The first estimate of (4.6) follows from relations (4.7) and (4.9).

In order to prove the second equality, we will derive equations for s_j^+ from equalities (2.3) and (4.5). Writing them in the form

$$\dot s_j^+ = (n_j - \dot w/(4w))s_j^+ + O(\varepsilon^4), \quad j = 1, 2$$

we replace s_j^+ on the right-hand sides of these equations by the expressions $s_j^+ = \tilde s_j^+ + O(\varepsilon)$, which follow from the first relation of (4.6). Taking the estimates $n_j = O(\varepsilon^2)$, $\dot w/(4w) = O(\varepsilon^3)$ into account, we obtain $\dot s_j^+ = (n_j - \dot w/(4w))\tilde s_j^+ + O(\varepsilon^3)$, which is equivalent to the second relation of (4.6).

Subtracting equality (4.3) from (4.5) and using the first relation of (4.6), we obtain an estimate of the error of the approximate quasisolution (4.3)

$$\Omega(t, \varepsilon) - \tilde\Omega^+(t, \varepsilon) = O(\varepsilon), \quad \Delta(t, \varepsilon) - \tilde\Delta^+(t, \varepsilon) = O(\varepsilon); \quad t \in [t_0, t_1]$$

We will denote by $\tilde\Omega$, $\tilde\Delta(t, \varepsilon)$ the approximate solution of Eqs. (2.3), which is obtained from equalities (4.3) by replacing the functions $\lambda_j^+(t, \varepsilon)$, defined in terms of $\xi^{(5)}$, α, β (t, ε), by the functions $\lambda_j(t, \varepsilon)$, which depend on t solely through $\xi^{(3)}$ (t, ε). It follows from relations (3.5) that $\lambda_j^+ - \lambda_j = O(\varepsilon^3)$, and hence the approximate solution $\tilde\Omega$, $\tilde\Delta(t, \varepsilon)$ has the same error $O(\varepsilon)$ as the approximate quasisolution (4.3).

Methods of constructing the asymptotic of the solutions of the equation $\ddot y + \lambda^2\omega^2(t)y = 0$ as $\lambda \to \infty$ have been described earlier.[6] To do this we also used a conversion of the equation into a quasidiagonal system.[7] Unlike these approaches,[6,7] in the present paper the coefficients of the linear part of Eqs. (2.3) depend on t via the variables ξ, which satisfy system of differential equations (2.2), the right-hand sides of which contain the required variables α and β.

For the L-system, the approximate quasisolution $\tilde\Omega_L$, $\tilde\Delta_L(t, \varepsilon)$ of the equations of angular motion is determined by formulae of the form (4.3) and (4.4) with the same constants C_j^+, but instead of $\lambda_j^+(t, \varepsilon)$, $k(t, \varepsilon)$, etc. we use the functions $\lambda_{jL}^+(t, \varepsilon)$, $k_L(t, \varepsilon)$, etc., calculated on the solution of the L-system considered.

From the fact that $\xi(t, \varepsilon) \in \Xi$ while Ω, $\Delta(t, \varepsilon) = O(1)$ when $t \in [t_0, t_1]$ is does not follow that the solution of the L-system possesses the same properties. In order to overcome this difficulty, we will consider the parallelepiped Ξ_ε which contains Ξ and components of the vector $\xi^{(5)}$ close to it:

$$\Xi_\varepsilon = \{\xi : (-\varepsilon, -\varepsilon, -1, \sqrt{\varepsilon}(C_{\upsilon*} - \varepsilon), -C_\theta^* - \varepsilon, -1, C_{p*} - \varepsilon) \le \xi \le$$
$$\le (C_x^* + \varepsilon, C_y^* + \varepsilon, 1, C_\upsilon^* + \varepsilon, C_\theta^* + \varepsilon, 1, C_p^* + \varepsilon)\} \tag{4.14}$$

Since $\xi_L(t_0, \varepsilon) = \xi(t_0, \varepsilon) \in \Xi$, a time interval $[t_0, t'] \subseteq [t_0, t_1]$ exists in which $\xi_L(t, \varepsilon) \in \Xi_\varepsilon$, $||\xi(t, \varepsilon) - \xi_L(t, \varepsilon)|| < \varepsilon$. As a result of the last inequality, it follows from the fact that the conditions for the flight to be accurate are satisfied when $\xi = \xi(t, \varepsilon)$ that they are satisfied when $\xi = \xi_L(t, \varepsilon)$. Hence, Ω_L, Δ_L $(t, \varepsilon) = O(1)$ when $t \in [t_0, t']$.

In the linearized equations of angular motion, we will introduce the complex variables $s_{jL}^+ (j = 1, 2)$ using formulae which differ from (4.5) solely in the fact that instead of λ_j^+, k, e, d we use d_{jL}^+, k_L, e_L, d_L in them. Since $h_{\Omega L}$, $h_{\Delta L} = 0$ in formulae similar to (4.8), we will have

$h_{2jL} = 0$. Hence, when $t \in [t_0, t']$, instead of relations of (4.6) we will have

$$s^+_{jL}(t, \varepsilon) - \tilde{s}^+_{jL}(t, \varepsilon) = O(\varepsilon^2), \quad ds^+_{jL}(t, \varepsilon)/dt - d\tilde{s}^+_{jL}(t, \varepsilon)/dt = O(\varepsilon^4); \quad j = 1, 2 \tag{4.15}$$

5. An estimate of the error of the linearized equations

The error in determining the slow variables. We will represent the fifth equation of system (2.2) in the form

$$\dot{\theta} = -\varepsilon^4 \frac{g\cos\theta}{\upsilon\cos\varepsilon^2\psi} + \varepsilon^4 \frac{\upsilon}{m}[R_2^{(0)}(y, \upsilon)\alpha - R_3^{(0)}(y, \upsilon, p)\beta] + g_\theta(\xi, \alpha, \beta, \varepsilon) + h_\theta(\xi, \alpha, \beta, \varepsilon) \tag{5.1}$$

where

$$g_\theta = \varepsilon^4 \frac{\upsilon(1 - \cos\varepsilon^2\psi)}{m\cos\varepsilon^2\psi}[R_2^{(0)}\alpha - R_3^{(0)}\beta] \tag{5.2}$$

Since $g_\theta = O(\varepsilon^8)$, the order of the additional term $g_\theta + h_\theta$ in Eq. (5.1) remains equal to the order of h_θ, i.e., $O(\varepsilon^7)$, but the possibility then arises of using the variable Δ in the linear part of the fifth and sixth equations of (2.2) to write them in the form of a single complex equation

$$\dot{\theta} + i\varepsilon^2\dot{\psi} = \hat{f}(\xi^{(5)}, \varepsilon) + \hat{f}_\Delta(\xi^{(3)}, \varepsilon)\Delta + g_\theta(\xi, \alpha, \beta, \varepsilon) + \hat{h}(\xi^{(5)}, \alpha, \beta, \varepsilon)$$

When writing the fifth equation of system (2.2) in the form (5.1), the vector form (2.6) of this system becomes

$$\dot{\xi} = f(\xi, \varepsilon) + f_{\alpha 0}(\xi, \varepsilon)\alpha + f_{\beta 0}(\xi, \varepsilon)\beta + g_\xi(\xi, \alpha, \beta, \varepsilon) + h_\xi(\xi, \alpha, \beta, \varepsilon) \tag{5.3}$$

Here $f_{\alpha 0}, f_{\beta 0}$ (ξ, ε) are equal to f_α, f_β (ξ, ε), taken at $\psi = 0$, and in the vector g_ψ only the fifth component g_θ is non-zero. For the L-system the vector equation of translational motion and of longitudinal rotation is obtained from Eq. (5.3) by dropping the column h_ξ.

We will consider two vector identities, which are obtained on substituting the solution of the initial system and the L-system into Eq. (5.3) and the corresponding linearized equation. Subtracting the second identity from the first and integrating, we arrive at the following equality, which is identical at $t \in [t_0, t']$

$$\xi(t, \varepsilon) - \xi_L(t, \varepsilon) = h_0(t, \varepsilon) + h_1(t, \varepsilon) + h_2(t, \varepsilon) \tag{5.4}$$

The following vector integrals are denoted by h_0, h_1 and h_2

$$h_0(t, \varepsilon) = \int_{t_0}^{t} [g_\xi(\tau, \varepsilon) + h_\xi(\tau, \varepsilon) - g_{\xi L}(\tau, \varepsilon)]d\tau$$

$$h_1(t, \varepsilon) = \int_{t_0}^{t} [f(\xi^{(5)}(\tau, \varepsilon), \varepsilon) - f(\xi_L^{(5)}(\tau, \varepsilon), \varepsilon)]d\tau$$

$$h_2(t, \varepsilon) = \int_{t_0}^{t} [f_{\alpha 0}(\xi^{(3)}(\tau, \varepsilon), \varepsilon)\alpha(\tau, \varepsilon) + f_{\beta 0}(\xi^{(3)}(\tau, \varepsilon), \varepsilon)\beta(\tau, \varepsilon) -$$

$$- f_{\beta 0}(\xi_L^{(3)}(\tau, \varepsilon), \varepsilon)\alpha_L(\tau, \varepsilon) - f_{\beta 0}(\xi_L^{(3)}(\tau, \varepsilon), \varepsilon)\beta_L(\tau, \varepsilon)]d\tau$$

In the first of these, the functions g_ξ, h_ξ $(\xi, \alpha, \beta, \varepsilon)$ calculated in the first solution of the initial system (5.3), (2.3) are denoted by g_ξ, h_ξ (τ, ε), and the function g_ξ, $(\xi, \alpha, \beta, \varepsilon)$, calculated in the solution of the L-system considered, are denoted by $g_{\xi L}$ (τ, ε).

For these integrals the following inequalities hold when $t - t_0 = O(\varepsilon^{-3})$

$$\|h_0(t, \varepsilon)\| \le \varepsilon^3 M_0, \quad \|h_1(t, \varepsilon)\| \le \varepsilon^3 L_1 \int_{t_0}^{t} \|\xi^{(5)}(\tau, \varepsilon) - \xi_L^{(5)}(\tau, \varepsilon)\| d\tau$$

$$\|h_2(t, \varepsilon)\| \le \varepsilon^3 M_2 + \varepsilon^3 L_2 \int_{t_0}^{t} \|\xi^{(5)}(\tau, \varepsilon) - \xi_L^{(5)}(\tau, \varepsilon)\| d\tau \tag{5.5}$$

where M_0, M_2, L_1 and $L_2 > 0$ are constants.

In order to obtain the first of inequalities (5.5), it is sufficient to note that $\|g_\xi\|$, $\|g_{\xi L}\| = O(\varepsilon^8)$, $\|h_\xi\| = O(\varepsilon^6)$ according to relations (5.2) and (2.5). The second equality follows from the fact that the vector function f in Eq. (5.3) has a Lipschitz constant in Ξ_ε of order ε^3.

To prove the third inequality of (5.5) consider the complex integral

$$\hat{h}_2(t, \varepsilon) = \int_{t_0}^{t} [\hat{f}_\Delta(\xi^{(3)}(\tau, \varepsilon), \varepsilon)\Delta(\tau, \varepsilon) - \hat{f}_\Delta(\xi_L^{(3)}(\tau, \varepsilon), \varepsilon)\Delta_L(\tau, \varepsilon)]d\tau$$

where

$$\hat{f}_\Delta = \varepsilon^4 \upsilon[R_2^{(0)} + iR_3^{(0)}]/m$$

Here we represent Δ and Δ_L by formulae of the form (4.5). To estimate the integrals that then occur with s_j^+, $s_{jL}^+ (j = 1, 2)$ we must use integration by parts, separating the differentials of the exponential functions $\exp i\varphi_j$, $\exp i\varphi_{jL}$ and using estimates (4.6) and (4.15). It is also necessary to take into account that $\omega_1^{-1} = O^*(1)$, $\omega_2^{-1} = O(\varepsilon^{-1})$, according to estimates (3.6). As a result we obtain the estimate

$$|\hat{h}_2(t,\varepsilon)| \le \varepsilon^3 \hat{M}_2 + \varepsilon^3 \hat{L}_2 \int_{t_0}^{t} \|\xi^{(5)}(\tau,\varepsilon) - \xi_L^{(5)}(\tau,\varepsilon)\| d\tau$$

where $\hat{M}_2$ and $\hat{L}_2 > 0$ are constants. Since $\|h^{(2)}\| \le |\hat{h}^{(2)}|$, the last inequality of (5.5) follows from this estimate.

When $t \in [t_0, t']$ we obtain the following inequality from inequalities (5.4) and (5.5)

$$\|\xi(t,\varepsilon) - \xi_L(t,\varepsilon)\| \le \varepsilon^3 M + \varepsilon^3 L \int_{t_0}^{t} \|\xi(\tau,\varepsilon) - \xi_L(\tau,\varepsilon)\| d\tau$$

where $M, L > 0$ are constants. Using Gronwall's lemma,[8] we obtain, when $t' \in [t_0, t']$, $t' - t^0 = O(\varepsilon^{-3})$, the required estimate

$$\|\xi(t,\varepsilon) - \xi_L(t,\varepsilon)\| \le \varepsilon^3 M \exp \varepsilon^3 L(t - t_0) = O_+(\varepsilon^3) \tag{5.6}$$

According to the procedure for deriving estimate (5.6) it holds in the section $[t_0, t'] \subseteq [t_0, t_1]$, where

$$\xi, \xi_L(t,\varepsilon) \in \Xi_\varepsilon; \quad \|\xi(t,\varepsilon) - \xi_L(t,\varepsilon)\| = O_+(\varepsilon) \tag{5.7}$$

We will assume that $t' < t_1$, so that when $t \in t', t_1$ relations (5.7) do not occur. Then, at the instant t′ one of two assumptions is satisfied: 1) the point $\xi_L(t', \varepsilon)$ belongs to the boundary Ξ_ε, while the order $\|\xi - \xi_L\|$ when $t = t'$ is either higher or equal to ε, 2) the point $\xi_L(t', \varepsilon)$ lies in Ξ_ε but the order $\|\xi - \xi_L\|$ when $t = t'$ is strictly equal to ε. The second of these assumptions contradicts estimate (5.6), which, as proved, is satisfied at the instant t'. However, the first assumption also leads to a contradiction.

Really the boundaries of the closed regions Ξ, Ξ_ε, defined by formulae (2.1) and (4.14), are parts of hyperplanes, orthogonal to the axes of coordinates. Here the region Ξ_ε is constructed in such a way that the distance from any point of its boundary to the region Ξ is measured along the corresponding coordinate and is greater than $O_+(\varepsilon^3)$. Hence, in the norm $\|\cdot\|$ the distance from the point $\xi_L(t', \varepsilon)$, which lies, by our assumption, on the boundary of Ξ_ε, to the point $\xi(t',\varepsilon)$, which lies in Ξ, is greater than $O_+(\varepsilon^3)$. This contradicts estimate (5.6), which at the instant t' is necessarily satisfied.

Thus, not one of the assumptions made is impossible, and consequently, conditions (5.7) are satisfied in the whole section $[t_0, t_1]$. Hence, estimate (5.6) holds over the whole section $[t_0, t_1]$.

The error in determining the fast variables. In order to obtain the error in determining the angular motion of the linearized system, we will represent the solutions Ω, $\Delta(t,\varepsilon)$ and Ω_L, $\Delta_L(t,\varepsilon)$ of the non-linear and linearized equations of motion by expressions of the form (4.5). By the estimates (3.6) we have $w = W$, $n_j = \varepsilon^2 N_j (j = 1, 2)$, where

$$W = O^*(1), \quad N_j = O(1), \quad \frac{\partial W}{\partial \xi^{(3)}} = O(1), \quad \frac{\partial N_j}{\partial \xi^{(3)}} = O(1) \text{ в } \Xi^{(3)}$$

Hence, it follows from estimate (5.6) that

$$w = w_L + O(\varepsilon^3), \quad n_j = n_{jL} + O(\varepsilon^5)$$

Then, by formula (4.4) for $\tilde{s}_j^+$ and the similar formula for $\tilde{s}_{jL}^+$ we have $\tilde{s}_j^+ - \tilde{s}_{jL}^+ = O(\varepsilon^2)$. The last equality, together with relations (4.6) and (4.15), leads to the estimate

$$s_j^+(t,\varepsilon) - s_{jL}^+(t,\varepsilon) = O(\varepsilon), \quad t \in [t_0, t_1], \quad j = 1, 2 \tag{5.8}$$

Further, according to equalities (4.10), we have

$$\partial e/\partial \xi^{(5)} = O(\varepsilon), \quad \partial d/\partial \xi^{(5)} = O(1) \text{ в } \Xi^{(5)}$$

Hence, from relations (4.10) and (5.6) we have the estimates

$$e(t,\varepsilon) - e_L(t,\varepsilon) = O(\varepsilon^4), \quad d(t,\varepsilon) - d_L(t,\varepsilon) = O(\varepsilon^3), \quad t \in [t_0, t_1] \tag{5.9}$$

Since, according to estimates (3.6), $\boldsymbol{\omega}_j = O(1)$, $\partial\omega_j/\partial\xi^{(3)} = O(1)$, in $\Xi^{(3)}$ we have $\omega_j = \omega_{jL} + O(\varepsilon^3)$. Then, from definition (4.4) of the phases φ_j and the similar definition of φ_{jL}, taking into account estimate (5.6), it follows that, when $t_1 - t_0 = O(\varepsilon^3)$, we will have

$$\varphi_j(t,\varepsilon) - \varphi_{jL}(t,\varepsilon) = O(1); \quad t \in [t_0, t_1], \quad j = 1, 2 \tag{5.10}$$

6. Conclusions

For the L-system (the system of equations of motion of the projectile, linearized with respect to the variables q, r, α and β) we have obtained estimates of the error of its solution compared with the solution of the initial non-linear system (2.2), (2.3) for the same initial conditions. They are expressed by relations (5.6) and (5.8)–(5.10). According to these estimates the L-system determines the variables x, y, z, ν, θ, ψ, p, the amplitude characteristics s_1^+, s_2^+ of the oscillations with respect to the variables $\Omega = q + ir$, $\Delta = \alpha + i\beta$ and the values of e and d, about which these oscillations occur, with a small error. At the same time, the L-system determines the phases φ_1 and φ_2 of the angular oscillations and, together with them, the variables Ω and Δ also, with a large error (of the order of the maximum values of the moduli of these variables).

With the assumptions made, the estimates (5.6), (5.8) and (5.9) are exact in order of magnitude in the sense that they are reached on certain trajectories. However, the error of the linearized equations may be reduced for examples when the flight time of the projectile is reduced or when the angular oscillations of its axis of symmetry are damped.

Since a decimal numerical scale was used when introducing the small parameter, the estimates obtained with respect to ε correspond to definite numerical estimates for the initial variables. For the coordinates x, y, z of the centre of mass they establish an error of the order of tens of meters for a shooting range of the order of tens of kilometres.

If the equations of motion of the projectile are linearized not only with respect to the angular motion variables q, r, α and β, but also with respect to the variable ψ, the orders of the additional non-linear terms in (5.3) and (2.3) do not change. Hence, for the system linearized in this way the estimates of the error (5.6) and (5.8)–(5.10), obtained for the L-system, remain true.

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